High-order perturbation expansion for the spectral analysis of fluid-loaded vibrating structure

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Summary

This paper deals with vibroacoustic under heavy fluid-loading conditions. The aim is to show how the high-order perturbation expansion approach described yields a very simple approximation of the spectrum of a heavy fluid-loaded structure. When the fluid-loading is "light" (e.g. metallic structures in contact with air) perturbation methods have been classically used to compute the spectrum of a fluid-loaded structure. Because of the computational efficiency of the expansions involved, it is now necessary to extend this method to heavy loading problems, such as those involving a metallic structure in contact with water. As we will see, although direct high order expansion methods yields unrealistic results, a judicious re-ordering of the various terms leads to surprisingly simple and efficient analytical results. Numerical examples are given in the simple case of a clamped rectangular plate under various fluid-loading conditions.

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1 Introduction

Accurately describing the effects of fluid loading on a vibrating structure leads to an overwhelming decrease in the computational efficiency of the various method available. This is due to the fact that the operator describing the loading is not only frequency dependent but also non local. The numerical methods used to compute the solution *in vacuo* generally leads to the resolution of linear systems with either small but frequency dependent matrices (as in the case of Boundary Element Methods) or large but sparse and frequency independent matrices (as in that of Finite Element Methods). When the loading is taken into account, the size of the involved matrices increase considerably, becoming full and frequency dependent. Any method reducing these drawbacks is therefore most welcome.

A great amount of effort has gone into developing suitable methods for this purpose. Generally, in each method one tries to remove or to approximate the frequency dependence of the fluid-loading operator. One of the most recent and popular of these is the space state method proposed by J.A. Giordano and G.H. Koopman [6, 14]. This method is based on the coupling between the Boundary Element Methods and Finite Element Methods, used to discretize the acoustic and structural domains respectively, in a canonical state space which takes in the form of a standard eigenvalue problem, where the frequency dependence of the loading matrix is removed by using low frequency approximations. While this method can be applied to heavy loading problems (such as those involving a metal structure in water), it is restricted to the low frequency range. An iterative method, also based on the coupling between the Boundary and Finite Element Methods, had been proposed by McCollum and Siders [11] to perform modal analysis on heavily fluid-loaded structures. It is based on the eigen-analysis of the structure *in vacuo*. As we already noticed, the principal difficulty induced by the fluid-loading is caused by the frequency dependence of the matrix elements describing it. Then at each step, the frequency dependence of these elements is fixed. At the first step, for each *in vacuo* mode, the fluid-loaded system is solved by using its eigenfrequency to compute the matrix elements describing the loading. This allows to solve a standard eigenvalues problem; which gives a first estimate of the fluid-loaded resonance mode and frequency. The next step uses these values and the procedure is repeated until convergence. But, as noted by [13], for a certain range of parameters this iterative scheme can give incorrect results; it is also very time consuming since it must be repeated for each mode under consideration.

Asymptotic analysis can be conduced when the loading is light, in the case of a metal structure in air, for instance. It consists in introducing a parameter ϵ, which is the ratio between the surface mass density and the fluid density, which is low. Using perturbation expansions [3, 4, 12], one can then construct an approximate solution, based on the *in vacuo* eigenmodes, which is not only easier to calculate than the solution of the exact problem but also leads to a better understanding of the various phenomenons involved. The main limitation here is that this method has a rather small validity range which it is difficult to estimate.

In this paper, we show how a simple modification of the usual high-order perturbation expansion methods yields a very simple approximation of the eigenmodes and resonance modes of a heavy fluid-loaded structure.

The resonance modes correspond to the free oscillations of the system and provide a natural tool for studying the effects of damping on sound radiation. The concept of eigenmodes and resonance modes for fluid-loading structure had been developed recently [3, 5]. In the literature, the resonance modes have been referred to under numerous denominations. The terms most commonly used are "generalized eigenmodes" and "complex modes". The latter denominations reflect the mathematical nature of these modes: they are a generalization of the eigenmodes of a mechanical system in which the frequency dependence of the various operators (such as the stiffness and mass operators) gives rise to complex mode values. The resonance modes and eigenmodes are different when frequency dependent damping is taken into account. This occurs in the case of material damping (in viscoelasticity), structural damping (in boundary dissipation problems) and in acoustic radiation problems.

Because of the frequency dependence, estimating the resonance modes is a much more difficult task than that involving eigenmodes.

In section 2 we present the equations. Section 3 deals with a high-order perturbation expansion of both eigenmodes and resonance modes. It is shown here how by re-arranging the various terms in the eigenvalue expansion, a simple analytical expression for the eigenvalues is obtained. In section 4, some numerical examples are given to show the efficiency of the method proposed. Some comments have been made in the Conclusion

2 Statement of the problem

In this paper, we deal with harmonic time dependence $\exp(-\imath\omega t)$, omitted to simplify the reading. Let us consider an infinite domain Ω with a finite or infinite boundary $\partial\Omega$ that contains a fluid at rest. A thin finite structure with thickness h and density ρ_p occupies a finite surface Σ of $\partial\Omega$. M is taken to denote a point on the structure and Q a point in the infinite domain. Let us take by $U(M,\omega)$ to denote the normal displacement of the structure. $\mathcal{O}$ is the operator, independent of ω, that describes the elastic behavior of the structure, such as the bi-Laplacian in the Kirchhoff plate equation or the Donnell-Mushtari operator in the case of a thin shell. At the boundary $\partial\Sigma$ of the structure, one adds mechanical boundary conditions. $f(M,\omega)$ is the excitation force. A perfect fluid at rest with density ρ_f and sound speed c_f occupies Ω; the acoustic pressure, which is denoted by $P(Q,\omega)$, is governed by the Helmholtz equation. A Neumann condition is imposed on Σ. $U(M,\omega)$ and $P(Q,\omega)$ satisfy the following system of equations:

$$\mathcal{O}U(M,\omega) - \rho_p h\omega^2 U(M,\omega) = F(M,\omega) - P(M,\omega) \text{ in } \Sigma \tag{1}$$

$$\Delta P(Q,\omega) + \omega^2/c_f^2 P(Q,\omega) = 0 \text{ in } \partial\Omega \tag{2}$$

$$\text{Boundary condition for } U(M,\omega) \text{ on } \partial\Sigma \tag{3}$$

$$\partial_n P(Q,\omega) = 0 \text{ on } \partial\Omega - \Sigma \tag{4}$$

$$\lim_{M\in\Omega\to Q\in\Sigma} \partial_n P(Q,\omega) = -\rho_f \omega^2 U(M,\omega) \text{ on } \Sigma. \tag{5}$$

2.1 Integrodifferential equation for the displacement

This system of equations for U and P can be reduced to an integrodifferential equation for U by using Green's representation for the pressure. Let $G(Q-Q',\omega)$ be the Green's function for the domain Ω with a Neumann boundary condition on $\partial\Omega$. This function can be obtained analytically for baffled plates or shells and numerically for other configurations. Using Green's representation for the pressure, it is easy to show that the pressure inside Ω is given by the integral equation

$$P(Q,\omega) = \omega^2 \rho_f \int_\Sigma U(M',\omega) G(Q-M',\omega) \mathrm{d}M'. \tag{6}$$

Introducing this result into equation (1), one obtains the integrodifferential boundary value problem for the structure

$$\mathcal{O}U(M,\omega) - \rho_p h\omega^2 \left(U(M,\omega) - \epsilon \int_\Sigma U(M',\omega) G(M-M',\omega) \mathrm{d}M' \right) = F(M,\omega) \tag{7}$$

$$\text{Boundary condition for } U(M,\omega) \text{ on } \partial\Sigma \tag{8}$$

where $\epsilon = \rho_f/\rho_p h$ is a small parameter in the case of a metal structure in air.

2.2 Eigenmodes and resonance modes.

One now introduces [3] the weak form of the system given by equations (7, 8). It reads : find $U(M,\omega)$ satisfying the boundary condition such that for each $V(M,\omega)$ satisfying the boundary condition, the following equation holds

$$a(U,V) - \Lambda \left(\langle U,V \rangle - \epsilon \beta_\omega(U,V) \right) = \langle F,V \rangle \tag{9}$$

where $\Lambda = \rho_p h \omega^2$. $a(U,V)$ corresponds to the elastic (or potential) energy of the structure. $\langle U,V \rangle$ is the usual inner product $\langle U,V \rangle = \int_\Sigma U(M) V^*(M) \mathrm{d}M$, $\rho_p h\omega^2 \langle U,V \rangle$ is the kinetic energy. $\beta_\omega(U,V) =$

$\int_\Sigma \int_\Sigma U(M)G(M-M',\omega)V^*(M')\mathrm{d}M\mathrm{d}M'$ is the radiation impedance. Since in the general case $G(M-M',\omega)$ depends on the frequency $\beta_\omega(U,V)$ is obviously frequency dependent. $\rho_p h\omega^2\epsilon_p\beta_\omega(U,V)$ is the energy exchanged between the structure and the fluid. Because of the frequency dependence of $\beta_\omega(U,V)$, it is necessary to distinguish the eigenmodes from the resonance modes [3, 5]. It is just recall here the corresponding definitions and basic properties of the two kind of modes.

The eigenmodes $\tilde{U}_m(M,\omega)$ and the eigenpulsations $\tilde{\omega}_m(\omega)$ of this problem are the non zero solutions of

$$a(\tilde{U}_m,V) - \tilde{\Lambda}_m(\omega)\left(\langle\tilde{U}_m,V\rangle - \epsilon\beta_\omega(\tilde{U}_m,V)\right) = 0, \text{ for each } V(M). \tag{10}$$

The eigenpulsations are given by $\tilde{\omega}_m^2(\omega) = \tilde{\Lambda}_m(\omega)/\rho_p h$.

The resonance modes $\hat{U}_m(M)$ (or generalized eigenmodes or complex modes) and the resonance pulsations $\hat{\omega}_m$ are defined as the nontrivial, frequency independent solutions of

$$a(\tilde{U}_m,V) - \rho_p h\hat{\omega}_m^2\left(\langle\hat{U}_m,V\rangle - \epsilon\beta_{\hat{\omega}_m}(\hat{U}_m,V)\right) = 0, \text{ for each } V(M). \tag{11}$$

While the resonance modes are harder to compute than the eigenmodes, they only need to be calculated once. Moreover, there is a simple relation between the eigenmodes and the resonance modes. From the definitions of these two kinds of modes, it is easy to see that one has:

$$\hat{U}_m(M) = \tilde{U}_m(M,\hat{\omega}_m), \tag{12}$$

$$\hat{\omega}_m^2 = \tilde{\omega}_m^2(\hat{\omega}_m). \tag{13}$$

Equation (13) implies that the resonance pulsations are the fixed points of the eigenpulsations. The only difficulty in computing the resonance modes is how to compute the resonance frequencies. In the general case, discretizing of the problem leads to a system of simultaneous equations with a full complex matrix. The complex zeros of the determinant of this matrix are the resonance frequencies, and their estimation is generally a very difficult task because the coefficients of the matrix depend on the frequency, which leads to estimating the zeros of a non-linear non-convex function in the complex plane. Problems of this kind have not yet been given rigorous numerical treatment. Once the resonance frequencies are known, the resonance modes are estimated by performing a simple matrix complex eigenmode search. Nevertheless, in the particular case of small damping problems (internal of by radiation), perturbation expansions lead to a particularly simple expression for the eigenmodes and eigenfrequencies [4]. In the following paragraph, we show how to use a perturbation method in the case of heavy fluid-loading.

3 High-Order Perturbation expansions

Let us expand into a Rayleigh-Schrödinger perturbation series [12] both the eigenmodes $\tilde{U}_m$ and the eigenvalues $\tilde{\Lambda}_m$ in ϵ :

$$\tilde{U}_m(M) = \tilde{U}_m^{(0)}(M) + \epsilon \tilde{U}_m^{(1)}(M) + \cdots + \epsilon^s \tilde{U}_m^{(s)}(M) + \cdots \tag{14}$$

$$\tilde{\Lambda}_m = \tilde{\Lambda}_m^{(0)} + \epsilon \tilde{\Lambda}_m^{(1)} + \cdots + \epsilon^s \tilde{\Lambda}_m^{(s)} + \cdots \tag{15}$$

The Rayleigh-Schrödinger Method is one of the most powerful methods available to dealing with vibroacoustics problems involving low frequency dependent damping such as those arising in viscosity or sound radiation in light fluid. Substituting the two perturbation expansions into the weak formulation given by equation (10), collecting and equating the coefficients of equal power ϵ to zero yields:

$$\epsilon^0 : a(\tilde{U}_m^{(0)}, V) - \tilde{\Lambda}_m^{(0)} \langle \tilde{U}_m^{(0)}, V \rangle = 0, \tag{16}$$

$$\vdots$$

$$\epsilon^s : a(\tilde{U}_m^{(s)}, V) - \tilde{\Lambda}_m^{(0)} \langle \tilde{U}_m^{(s)}, V \rangle = \sum_{l=1}^{l=s} \tilde{\Lambda}_m^{(l)} \langle \tilde{U}_m^{(s-l)}, V \rangle - \sum_{l=0}^{l=s-1} \tilde{\Lambda}_m^{(l)} \beta_\omega (\tilde{U}_m^{(s-l-1)}, V), \tag{17}$$

Equation (16) is the usual relation giving the eigenmodes of the elastic structure *in vacuo.* If the eigenmodes $\tilde{U}_m^{(0)}(M)$ of the elastic plate *in vacuo* are normalized, it is easy to obtain the first order term of the eingenvalue, that is

$$\frac{\tilde{\Lambda}_m^{(1)}}{\tilde{\Lambda}_m^{(0)}} = \beta_\omega^{mm}, \tag{18}$$

in the previous equation, it has been denoted $\beta_\omega^{mn} = \beta_\omega(\tilde{U}_m^{(0)}, \tilde{U}_n^{(0)*})$. To simplify the reading this notation will be used in the rest of the paper. $\tilde{U}_m^{(1)}(M)$ is expanded into a series of zeroth-order modes: $\tilde{U}_m^{(1)}(M) = \sum_n \alpha_m^{1n} \tilde{U}_n^{(0)}(M)$ where the coefficients α_m^{1n} are easy to calculate [12]. For all $s > 1$, one has:

$$\tilde{\Lambda}_m^{(s)} = \sum_{l=0}^{l=s-1} \tilde{\Lambda}_m^{(l)} \beta_\omega(\tilde{U}_m^{(s-l-1)}, \tilde{U}_m^{(0)}) - \sum_{l=1}^{l=s-1} \tilde{\Lambda}_m^{(l)} \langle \tilde{U}_m^{(s-l)}, \tilde{U}_m^{(0)} \rangle. \tag{19}$$

In a similar manner, each component of the eigenmode's perturbation expansion is developed into a series of zeroth-order modes:

$$\tilde{U}_m^{(s)}(M) = \sum_n \alpha_m^{sn} \tilde{U}_n^{(0)}(M). \tag{20}$$

One obtains, for the eigenvalues:

$$\frac{\tilde{\Lambda}_m^{(2)}}{\tilde{\Lambda}_m^{(0)}} = (\beta_\omega^{mm})^2 + \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp}, \tag{21}$$

$$\frac{\tilde{\Lambda}_m^{(3)}}{\tilde{\Lambda}_m^{(0)}} = (\beta_\omega^{mm})^3 + 2\beta_\omega^{mm} \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} + \sum_{p \neq m} \alpha_m^{2p} \beta_\omega^{mp}, \tag{22}$$

$$\begin{aligned} \frac{\tilde{\Lambda}_m^{(4)}}{\tilde{\Lambda}_m^{(0)}} = {} & (\beta_\omega^{mm})^4 + 3(\beta_\omega^{mm})^2 \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} + 2\beta_\omega^{mm} \sum_{p \neq m} \alpha_m^{2p} \beta_\omega^{mp} + \sum_{p \neq m} \alpha_m^{3p} \beta_\omega^{mp} \\ & + \left(\sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} \right)^2 - \alpha_m^{2m} \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp}, \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\tilde{\Lambda}_m^{(5)}}{\tilde{\Lambda}_m^{(0)}} = {} & (\beta_\omega^{mm})^5 + 4(\beta_\omega^{mm})^3 \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} + 3(\beta_\omega^{mm})^2 \sum_{p \neq m} \alpha_m^{2p} \beta_\omega^{mp} + 2\beta_\omega^{mm} \sum_{p \neq m} \alpha_m^{3p} \beta_\omega^{mp} \\ & + 3\beta_\omega^{mm} \left(\sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} \right)^2 - 2\beta_\omega^{mm} \alpha_m^{2m} \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} \\ & + \sum_{p \neq m} \alpha_m^{4p} \beta_\omega^{mp} + 2 \sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp} \sum_{p \neq m} \alpha_m^{2p} \beta_\omega^{mp} - \sum_{p \neq m} \alpha_m^{1p} \alpha_m^{3p} \beta_\omega^{mp} - \alpha_m^{2m} \sum_{p \neq m} \alpha_m^{2p} \beta_\omega^{mp}. \end{aligned} \tag{24}$$

Now let us calculate the eigenvalue expansion up to order 5:

$$\frac{\tilde{\Lambda}_m}{\tilde{\Lambda}_m^{(0)}} = 1 + \sum_{s=1}^{s=5} \epsilon^s \frac{\tilde{\Lambda}_m^{(s)}}{\tilde{\Lambda}_m^{(0)}} + \cdots + \cdots. \tag{25}$$

Upon re-ordering the terms, one obtains:

$$\begin{aligned}
\frac{\tilde{\Lambda}_m}{\tilde{\Lambda}_m^{(0)}} = & \left[1 + \epsilon\beta_\omega^{mm} + \epsilon^2(\beta_\omega^{mm})^2 + \epsilon^3(\beta_\omega^{mm})^3 + \epsilon^4(\beta_\omega^{mm})^4 + \epsilon^5(\beta_\omega^{mm})^5 + \cdots\right] \\
& + \epsilon^2 \sum_{p\neq m} \alpha_m^{1p}\beta_\omega^{mp}\left[1 + 2\epsilon\beta_\omega^{mm} + 3\epsilon^2(\beta_\omega^{mm})^2 + 4\epsilon^3(\beta_\omega^{mm})^3 + \cdots\right] \\
& + \epsilon^3 \sum_{p\neq m} \alpha_m^{2p}\beta_\omega^{mp}\left[1 + 2\epsilon\beta_\omega^{mm} + 3\epsilon^2(\beta_\omega^{mm})^2 + \cdots\right] \\
& + \epsilon^4 \left\{ \sum_{p\neq m} \alpha_m^{3p}\beta_\omega^{mp}\left[1 + 2\epsilon\beta_\omega^{mm} + \cdots\right] + \left(\sum_{p\neq m} \alpha_m^{1p}\beta_\omega^{mp}\right)^2 \left[1 + 3\epsilon\beta_\omega^{mm} + \cdots\right] \right. \\
& \qquad \left. - \alpha_m^{2m} \sum_{p\neq m} \alpha_m^{1p}\beta_\omega^{mp}\left[1 + 2\epsilon\beta_\omega^{mm} + \cdots\right] \right\} \\
& + \epsilon^5 \cdots
\end{aligned} \tag{26}$$

For $\epsilon\beta_\omega^{mm} < 1$, one can recognize in each square bracket the expansion of $1/(1-\epsilon\beta_\omega^{mm})$ and its successive powers. By identifying the expansion with the original functions, one obtains the modified high order perturbation expansion:

$$\begin{aligned}
\frac{\tilde{\Lambda}_m}{\tilde{\Lambda}_m^{(0)}} = & \frac{1}{(1-\epsilon\beta_\omega^{mm})} + \epsilon^2 \frac{\sum_{p\neq m} \alpha_m^{1p}\beta_\omega^{mp}}{(1-\epsilon\beta_\omega^{mm})^2} + \epsilon^3 \frac{\sum_{p\neq m} \alpha_m^{2p}\beta_\omega^{mp}}{(1-\epsilon\beta_\omega^{mm})^2} \\
& + \epsilon^4 \left\{ \frac{\sum_{p\neq m} \alpha_m^{3p}\beta_\omega^{mp}}{(1-\epsilon\beta_\omega^{mm})^2} + \frac{\left(\sum_{p\neq m} \alpha_m^{1p}\beta_\omega^{mp}\right)^2}{(1-\epsilon\beta_\omega^{mm})^3} - \alpha_m^{2m} \frac{\sum_{p\neq m} \alpha_m^{1p}\beta_\omega^{mp}}{(1-\epsilon\beta_\omega^{mm})^2} \right\} \\
& + \epsilon^5 \cdots
\end{aligned} \tag{27}$$

In some particular cases (see Appendix), this method gives exact results and can be viewed as an analytic continuation process. If all the terms β_ω^{mp} are zero for $m \neq p$, the first order of the modified perturbation expansion leads to exact results, even if $\epsilon\beta_\omega^{mm}$ is greater than 1. This result also shows that the true convergence condition of the various perturbation series is not simply $\epsilon < 1$ but $\epsilon\beta_\omega^{mm} < 1$. It is worth noting that this makes the numerical analysis almost impossible because the convergence condition is different for every mode and depends on the frequency.

4 Numerical Results

In this section, numerical examples are given for baffled fluid-loaded clamped rectangular plates under heavy loading conditions. This plate occupies a domain of the $z = 0$ plane in $I\!R^3$ defined by $x \in]0, a[\times y \in]0, b[$. The same fluid occupies the two half-space $x > 0$ and $x < 0$. Let us recall that in this case, the Green's function for the Neumann problem of the Helmholtz equation is given by:

$$G(M - M', \omega) = \frac{-1}{4\pi} \left(\frac{\mathrm{e}^{\imath \frac{\omega}{c_f} d(M,M')}}{d(M, M')} + \frac{\mathrm{e}^{\imath \frac{\omega}{c_f} d(M,M'')}}{d(M, M'')} \right)$$

where $d(M, M')$ is the Euclidian distance between M and M', M'' is the symmetric of M' with respect to the $z = 0$ plane. Then, the only numerical difficulty arising with this method lies in the computation of the various modal acoustic impedance β_ω^{mnpq} given by the integral, with $X = ax$ and $Y = by$:

$$\beta_\omega^{mnpq} = a^2 b^2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 U_{mn}^{(0)}(x, y) G(x - x', y - y', \omega) U_{pq}^{(0)}(x', y') \mathrm{d}x \mathrm{d}x' \mathrm{d}y \mathrm{d}y' \tag{28}$$

The evaluation of this kind of integral is considerably simplified when the eigenmodes are given in separate form $U_{mn}^{(0)}(x, x) = X_m(x) Y_n(y)$ by a linear transformation in the integral domain given by $X = x - x'$ and $Y = y - y'$ [9]. One thus obtains :

$$\beta_\omega^{mnpq} = 4a^2 b^2 \int_0^1 \int_0^1 A_{mp}(X) B_{nq}(Y) G(X, Y, \omega) \mathrm{d}X \mathrm{d}Y, \tag{29}$$

$$A_{mp}(X) = \int_0^{1-X} X_m(X + x') X_p(x') \mathrm{d}x', \;\; B_{nq}(Y) = \int_0^{1-Y} Y_m(Y + y') Y_p(y') \mathrm{d}y'. \tag{30}$$

In many cases, such as the one studied here, simple analytic expressions are available for the functions $A_{mp}(X)$ and $B_{nq}(Y)$, which make it possible to compute the modal impedances by simply using double integrals.

In order to check the validity of the method, the spectrum of the structure is calculated using the original and modified perturbation expansions up to order 3 and compared with exact computations. The exact computations are based on a Chebychev-collocation approximation of the boundary integral equations associated with the problem [10]. Roughly, the two unknowns in the problem, namely the displacement and the acoustic sound pressure jump across the plate, are the solutions of two coupled integral equations. These functions are approximated by series of Chebychev orthogonal polynomials of the first kind. Using a collocation method, the new unknowns (the coefficients of the two series) are the solutions of a linear system of simultaneous equations. The complex zeros of the determinant of the matrix of this system are the resonance frequencies. These frequencies are computed using a Newton-Raphson method.

For each mode, denoted by its index mn, the resonance spectrum is calculated by solving the relation 13: $\hat{\omega}_{mn}^2 = \tilde{\omega}_{mn}^2(\hat{\omega}_{mn}) = \tilde{\Lambda}_{mn}(\hat{\omega}_{mn})/\rho_p h$. For each mode, there are two frequencies that satisfy this relation, namely $\hat{\omega}_{mn}$ and $-\hat{\omega}_{mn}^*$. These frequencies are estimated using the approximate expressions for the eigenvalues $\tilde{\Lambda}_{mn}(\omega)$ given by the direct perturbation expansion (26) or by the modified perturbation expansion (27) truncated at orders ranging from 1 to 3. The roots $\hat{\omega}_{mn}$ of the equation $\omega^2 = \tilde{\omega}_{mn}^2(\omega)$ were obtained using the Mathematica [16] root finding routine without any particular difficulty.

With this method, analytical or numerical knowledge about the *in vacuo* eigenmodes is necessary. In the case of a clamped rectangular plate, there are no exact analytical results available, such as those on a simply supported plate, but the analytical approximation by Warburton [8] gives a quite perfect shape for the eigenmodes and only slightly overestimates the eigenfrequencies by about 3%.

The first results are given in table 1. Four different configurations were tested in water (in air the differences between the exact and approximate solutions are negligible). The plate is rectangular having length $a = 1m$, width $b = 0.7m$ and varying thickness h. It is made of steel with Young's modulus $E = 200$GPa, Poisson coefficient $\nu = 0.3$ and density $\rho_p = 7800$Kg/m^3. The fluid is water with density $\rho_f = 1000$Kg/m^3 and sound celerity $c_f = 1500$m/s. Table 1 gives the data obtained on steel plates in water, which is the usual heavy loading situation, with thickness of $5mm$, $1cm$, $3cm$ and $10cm$.

In this table, the first resonance frequencies of the plate under various fluid-loading conditions calculated up to the third order of the perturbation expansion are compared with the exact computations. In the exact values column, the fluid-loaded spectrum is given along with the *in vacuo* one. In order to clearly show the added mass corresponding to a decrease in the real part and the damping that give rise to a negative imaginary part. The seven remaining columns give the results obtained using the original and modified perturbation expansions. The zeroth order is the *in vacuo* spectrum (which is about 3% larger than the exact one because of the Warburton approximation). The order 1 of the perturbation expansion are the data obtained by solving equation (26) limited to the first order, while the order 1 modified data are those obtained by solving equation (27) to the first order, and so on up to the higher orders. All clearly divergent results (with a positive imaginary part) are given in italics.

The most important finding made in this study is that all the results show that the first order modified approximation gives very accurate results under heavy loading conditions with a small numerical cost. Some resonances are obviously less exactly described, such as mode $(m = 1, n = 3)$ in the case of the thinner plates. But even in these cases, the estimate given by the first order modified approximation is fairly accurate, or at least sufficient for the acoustician.

The next results, presented figures 1 and 2, show the evolution of the resonance frequency for the mode $(m = 1, n = 1)$ in figure 1 and for the mode $(m = 1, n = 3)$ in figure 2 when the thickness of the plate is varying. In both figures, the top curve shows the real part of the resonance and the bottom the imaginary part of it. The smooth curves are the results given by the modified approximation while the curves with the diamonds are the exact results. From these curves, it is obvious that again the modified approximations gives particularly good results.

More interesting is the phenomenon presented for the for the mode $(m = 1, n = 3)$. In figure 2 one observes two disjointed curves. The first one, starting from the origin (when the thickness of the plate tends to zero all resonance frequencies are nil) is called the lower branch and the other the upper

Table 1: Comparison of exact and approximate spectra for a steel plate in water (values given in Hz)

		Exact		Order of the perturbation expansion						
		in vacuo	fluid-loaded	0	1	1 (modified)	2	2 (modified)	3	3 (modified)
h=5mm	Mode 1 1	68.86	17.2 $-\imath$ 0.11	69.06	$0 +\imath\ 234$	17.3 $-\imath$ 0.11	$651.5 +\imath\ 281$	$0 +\imath\ 21.7$	$0 +\imath\ 1133.5$	$119.6 +\imath\ 14.3$
$\epsilon = 25.6$	Mode 1 2	166.05	61.87 $-\imath$ 0.0017	166.57	0. $-\imath$ 365.8	62.06 $-\imath$ 0.0017	$898 +\imath\ 458.1$	$172.5 +\imath\ 30.7$	$0 +\imath\ 1634$	$0 +\imath\ 400.9$
	Mode 1 3	313.32	135.01 $-\imath$ 0.82	314.1	$0 +\imath\ 576.2$	125.9 $-\imath$ 2.78	$1199 +\imath\ 629.1$	$278.9 +\imath\ 28.5$	$0 +\imath\ 2041.77$	$0 +\imath\ 565.5$
h=1cm	Mode 1 1	137.7	47.1 $-\imath$ 0.78	138.12	$0 +\imath\ 307.3$	47.6 $-\imath$ 0.77	$630.7 +\imath\ 270.3$	$0 +\imath\ 22.3$	$0 +\imath\ 882.2$	$158.1 +\imath\ 24.4$
$\epsilon = 13.8$	Mode 1 2	332.1	163.5 $-\imath$ 0.08	333.1	$0 +\imath\ 465.1$	164.0 $-\imath$ 0.07	$867 +\imath\ 453.9$	$298 +\imath\ 75.3$	$0 +\imath\ 1324$	$0 +\imath\ 437.4$
	Mode 1 3	626.6	347.6 $-\imath$ 4.9	628.2	$0 +\imath\ 669.2$	332.9 $-\imath$ 16.3	$1168 +\imath\ 605.1$	$513.3 +\imath\ 89.6$	$0 +\imath\ 1626.4$	$0 +\imath\ 595.9$
h=3cm	Mode 1 1	413.1	220.2 $-\imath$ 13.9	414.3	$0 +\imath\ 388$	222. $-\imath$ 13.6	$594.7 +\imath\ 220.9$	$180.4 +\imath\ 25.6$	$0 +\imath\ 483.3$	$276. +\imath\ 29.6$
$\epsilon = 4.3$	Mode 1 2	996.3	677.0 $-\imath$ 12.87	999.4	$0 +\imath\ 159.2$	679.9 $-\imath$ 11.9	$911.3 +\imath\ 389.4$	$701.9 +\imath\ 148.2$	$0 +\imath\ 780.5$	1181 $-\imath$ 23.9
	Mode 1 3	1879.9	1648.1 $-\imath$ 8.52	1884.7	1202.6 $-\imath$ 284	1453.7 $-\imath$ 60.8	$1480.3 +\imath\ 411.6$	$1404.7 +\imath\ 140.6$	$2150.3 +\imath\ 356.2$	1883.1 $-\imath$ 66.5
h=10cm	Mode 1 1	1377.1	1089.2 $-\imath$ 199.1	1381.2	1296.9 $-\imath$ 589.5	1093.6 $-\imath$ 195.4	1074.6 $-\imath$ 75.6	1122.6 $-\imath$ 230.2	1280.6 $-\imath$ 289.7	1075.5 $-\imath$ 239.9
$\epsilon = 1.3$	Mode 1 2	3321	3122.4 $-\imath$ 559.1	3331.5	3396.4 $-\imath$ 534.1	3137.5 $-\imath$ 570.6	2960.3 $-\imath$ 514.2	3143.5 $-\imath$ 465.3	3139.1 $-\imath$ 393.4	3157 $-\imath$ 451.1
	Mode 1 3	6266.4	6273.4 $-\imath$ 368.7	6282.2	6320.1 $-\imath$ 361.4	6279.5 $-\imath$ 368.7	6279.3 $-\imath$ 375.5	6277.5 $-\imath$ 370.3	6276.9 $-\imath$ 370.6	6277.5 $-\imath$ 370.2

branch. In the particular case considered here, except for modes $(m = 1, n = 1)$, $(m = 2, n = 1)$ and $(m = 1, n = 2)$, all the modes showed this phenomenon, which is a typical behavior of the non-linear systems. *A priori* this must not occurs in the present case since the system that one considers here do not present any non-linear classical feature such as structural non-linearity, non-linear propagation in the fluid or non linear moving boundary condition at the interface between the structure and the fluid [15]. The phenomena observed here has a strong link with the concept of frequency non-linear modes [2] which exhibit a non-linearity that depends on the temporal parameters rather than on the geometrical parameters such as propagation of sound in poroelastic media. The frequency non-linear modes are related to problems expressed in the following form $\mathcal{M}(\omega)u = f$ where u is the unknown, f the forcing term and $\mathcal{M}(\omega)$ is a non linear frequency dependent operator such as those that occur in poroelastic media or in fluid-structure interaction for which the loading operator described by equation 6 has a non-linear frequency dependence. In this latter case, when the fluid-loading is light, this non-linear dependence is not significant but when the fluid-loading is strong, this non-linearity can be observed as shown by the results presented here.

To be sure that this effect is not a numerical artifact, we have plotted in the figures (3) and (4) the amplitude of the displacement of the plate, given by the resolution of the Boundary Integral Equations associated with the exact solution, for frequencies close to the two resonances observed for the thickness corresponding to the inflexion point for the real parts and a crossing of the imaginary parts that is, for the $(m = 1, n = 3)$ mode close to $h = 0.074cm$. The plate is excited by a unit Dirac delta located close to the middle of the plate at $x = 0.6m$, $y = 0.45m$. The excitation frequency f, given in the figure caption, has been chosen close to (less than 0.1%) the resonance one; obviously at the resonance, the exact solution has no solution. In the figure (3), the excitation frequency $f = 4827 - \imath 1025\ Hz$ is close to the resonance frequency of the upper branch (that is $\hat{f}^u_{13} = 4829.64 - \imath 1024.4\ Hz$). In the figure (4), the excitation frequency $f = 3717 - \imath 1053 Hz$ is close to the resonance frequency to the lower branch (that is $\hat{f}^u_{13} = 3716.9 - \imath 1052.4\ Hz$). As shown by these figures, the displacement is very close to that of a $m = 1, n = 3$ mode. Similar results had been observed for many modes; more precisely, only the $(m = 1, n = 1)$, $(m = 1, n = 2)$ and $(m = 2, n = 1)$ modes vary continuously with the thickness, all the other modes presents such a comportment.

5 Conclusion

The main finding to emerge from this study was that the efficiency of the first order modified approximation does not depend on the loading, at least in the particular case consider here. The second important point is that the modified approximation can be obtained without any difficulty. Unlike the usual first order approximations, its estimation does not require any computational effort. With this method, it seems to be possible to describe all the features of the spectrum of a fluid-loaded structure, such as the high added mass and small damping, observed in water in the case of a the thinnest plate or conversely, the small added mass and high damping observed with the thickest plate. As long as one is able to estimate the eigenmodes of the *in vacuo* structure, using a finite element model, for example, and the loading given by Green's function for the Neumann problem in the Helmholtz equation, it is easy to compute the spectrum of a heavily fluid-loaded structure at a negligible numerical cost. This method obviously requires the *a priori* knowledge of this Green's function, that of an infinite plane screen or cylindrical surface for example. In most cases, however this function is impossible to compute analytically. It is therefore necessary to use either a numerical calculation of the Green function or an approximate description of the loading, for example based on the Rayleigh integral but the validity of an approximation of this kind still remains to be confirmed.

Some aspects of this method need to be studied in greater depth. The most obvious one is the the nature of this approximation, especially the effects of neglecting the modal coupling given by the terms $\sum_{p \neq m} \alpha_m^{1p} \beta_\omega^{mp}$. It is well known in damping theory studies that neglecting the modal coupling in non-proportional damped systems yields poor results, even when the structure is only slightly damped [1]. The author has no clear explanation of why the first order modified expansion works so well. More precisely, an expansion which neglects the intermodal impedance ought to work badly. All previous authors that have worked in this subject have suggested that the intermodal impedance plays a significant role in the loading, and that a method which neglects it must fail. But the results found here seem to invalidate this hypothesis, and it would be interesting to know if the hypothesis is also invalid in other geometries. The next point that needs to be studied is the apparition of a frequency non-linear mode caused by the fluid-loading. It must be deeply studied to understand its emergence in the case considered here and if this occurs for other geometries, boundary conditions or other type of structure (like shells or beams).

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A High order perturbation expansion of a viscous plate

In this Appendix, some classical results [7, 17] on the eigenmodes and resonance modes of a viscously damped plate are used to show that the high perturbation expansion is able to give the exact solution in some particular cases.

Let $U(x,\omega)$ be the normal displacement of a simply supported viscous plate in the harmonic regime with time dependence $\exp(-\imath\omega t)$. Let us recall that in the time domain, the simplest form of the viscous operator is given by $\eta_p\partial/\partial t$, where η_p is the damping constant. If we introduce the parameter $\epsilon_p = \eta_p/\rho_p h$, then $U(x,\omega)$ satisfies the modified Kirchhoff equation as well as the boundary conditions:

$$D\Delta^2 U(x,\omega) - \rho_p h\omega^2 \left(1 - \epsilon_p \frac{-\imath}{\omega}\right) U(x,\omega) = F(x,\omega) \tag{31}$$

$$U(0,\omega) = 0, U''(0,\omega) = 0, U(l_x,\omega) = 0, U''(L_x,\omega) = 0 \tag{32}$$

where $D = Eh^3/(12(1-\nu^2))$ is the bending rigidity of the plate.

A.1 Exact solution for the eigenmodes and resonance modes

Let us denote $\rho_p^{'} = \rho_p\left(1 - \epsilon_p \frac{-\imath}{\omega}\right)$. The eigenmodes $\tilde{U}_m(x,\omega)$ and eigenpulsations $\tilde{\omega}_m(\omega)$ are the non-trivial solution of:

$$D\Delta^2 \tilde{U}_m(x,\omega) - \rho_p^{'}(\omega) h\tilde{\omega}_m^2(\omega)\tilde{U}_m(x,\omega) = 0 \tag{33}$$

$$\tilde{U}_m(0,\omega)=0, \tilde{U}''_m(0,\omega)=0, \tilde{U}_m(l_x,\omega)=0, \tilde{U}''_m(L_x,\omega)=0 \tag{34}$$

Then, it is easy to show that one has:

$$\tilde{U}_m(x,\omega) = \sqrt{\frac{2}{L_x}}\sin\frac{m\pi}{L_x}x \tag{35}$$

$$\tilde{\omega}_m(\omega) = \left(\frac{m\pi}{L_x}\right)^2\sqrt{\frac{D}{\rho_p h}}\frac{1}{\sqrt{1+\epsilon_p\frac{\imath}{\omega}}} \tag{36}$$

The resonance modes $\hat{U}_m(x)$ and resonance pulsations $\hat{\omega}_m$ are the non-trivial solution of:

$$D\Delta^2\hat{U}_m(x)-\rho_p h\hat{\omega}_m^2\left(1-\epsilon_p\frac{-\imath}{\hat{\omega}_m}\right)\hat{U}_m(x)=0 \tag{37}$$

$$\hat{U}_m(0)=0, \hat{U}''_m(0)=0, \hat{U}_m(l_x)=0, \hat{U}''_m(L_x)=0 \tag{38}$$

The resonance modss are linked to the eigenmodes by the relations (12): $\hat{U}_m(x)=\tilde{U}_m(x,\hat{\omega}_m)$ and (13): $\hat{\omega}_m^2=\tilde{\omega}_m^2(\hat{\omega}_m)$. Since the eigenmodes do not depend on the frequency, the eigenmodes and resonance modes are identical. The resonances frequencies are the solutions of

$$\hat{\omega}_m^2=\tilde{\omega}_m^2(\hat{\omega}_m)=\left(\frac{m\pi}{L_x}\right)^4\frac{D}{\rho_p h}\frac{1}{1+\epsilon_p\frac{\imath}{\hat{\omega}_m}} \tag{39}$$

One then has to solve $\frac{\rho_p h}{D}\hat{\omega}_m^2+\frac{\imath\epsilon_p\rho_p h}{D}\hat{\omega}_m-\left(\frac{m\pi}{L_x}\right)^4=0$. This quadratic equation ca be easily solved, one obtains the two roots:

$$\hat{\omega}_m=-\frac{\imath\epsilon_p}{2}\pm\sqrt{\frac{D}{\rho_p h}\left(\frac{m\pi}{L_x}\right)^4-\left(\frac{\epsilon_p}{2}\right)^2} \tag{40}$$

This root symmetry with respect to the imaginary axis is the condition [5] for the resonance modes to ensure causality and provide real solutions with a time dependency of the form $\exp(-\imath\omega t)$. Obviously with a time dependency of the form $\exp(\imath\omega t)$, the imaginary parts of the roots have to be positive.

A.2 Perturbation expansion for the eigenmodes

The weak form of the equation giving the eigenmodes reads : find $\tilde{U}_m(x,\omega)$ and $\tilde{\Lambda}_m(\omega)$ satisfying the boundary condition such that for each $V(x,\omega)$ satisfying the boundary condition, the following equation holds

$$a(\tilde{U}_m,V)-\tilde{\Lambda}_m\left(\langle\tilde{U}_m,V\rangle-\epsilon_p\beta_\omega(\tilde{U}_m,V)\right)=0 \tag{41}$$

where $\beta_\omega(\tilde{U}_m, V) = -\imath/\omega\langle\tilde{U}_m, V\rangle$ and $\tilde{\Lambda}_m = \rho_p h\tilde{\omega}_m^2(\omega)$. It can be easily seen that one has $\beta_\omega^{mp} = -\imath/\omega\delta_m^p$, with δ_m^p the Kronecker delta.

Then, since the weak form of this equation is similar to that of the fluid-loaded plate 10, all the results previously obtained can be used without making any changes. It is easy to show that the zeroth order terms are, with $\tilde{\Lambda}_m^{(0)} = \rho_p h(\tilde{\omega}_m^{(0)})^2$, given by:

$$\tilde{U}_m^{(0)}(x) = \sqrt{\frac{2}{L_x}}\sin\frac{m\pi}{L_x}x, \langle\tilde{U}_m^{(0)}, \tilde{U}_m^{(0)*}\rangle = 1 \quad (42)$$

$$\tilde{\omega}_m^{(0)} = \left(\frac{m\pi}{L_x}\right)^2\sqrt{\frac{D}{\rho_p h}} \quad (43)$$

Then, because $\beta_\omega^{mp} = -\imath/\omega\delta_m^p$, the high order perturbation expansion of the eigenmodes and eigenvalues is given by

$$\tilde{U}_m(x,\omega) = \tilde{U}_m^{(0)}(x), \quad (44)$$

$$\tilde{\Lambda}_m(\omega) = \tilde{\Lambda}_m^{(0)}\frac{1}{1+\frac{\imath\epsilon_p}{\omega}},\ \tilde{\omega}_m(\omega) = \tilde{\omega}_m^{(0)}\frac{1}{\sqrt{1+\frac{\imath\epsilon_p}{\omega}}}, \quad (45)$$

which shows that the high order perturbation expansion of the eigenmodes leads in this simple case to the exact solution (see equation 39).

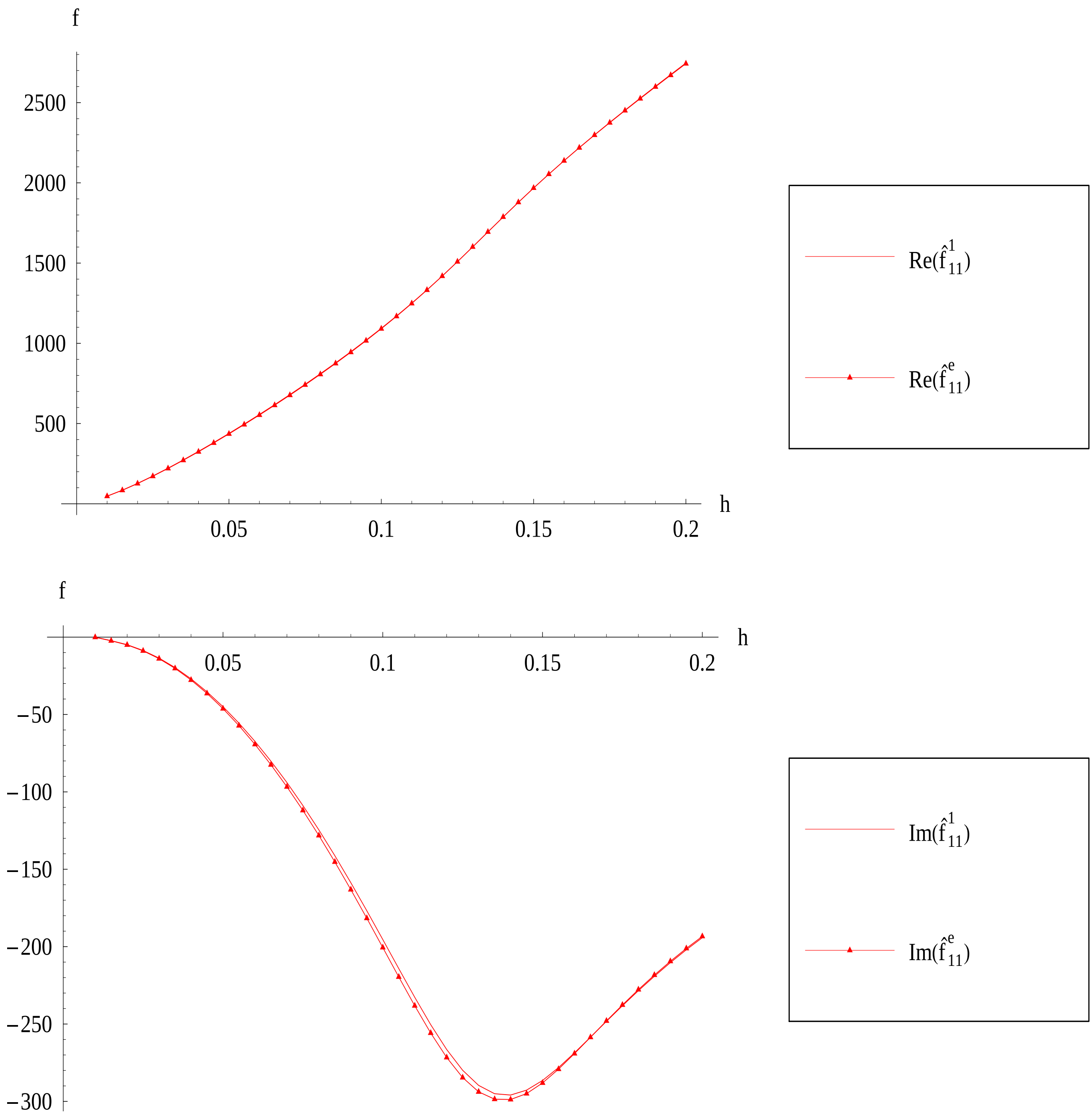

Figure 1: $1m \times 0.7m$ steel plate in water: evolution of the real and imaginary parts with the thickness of the plate of the $(m = 1, n = 1)$ resonance frequency. Comparison of the exact computations $(\hat{f}^e_{11})$ with the modified approximation $(\hat{f}^1_{11})$. Top: real part, bottom: imaginary part

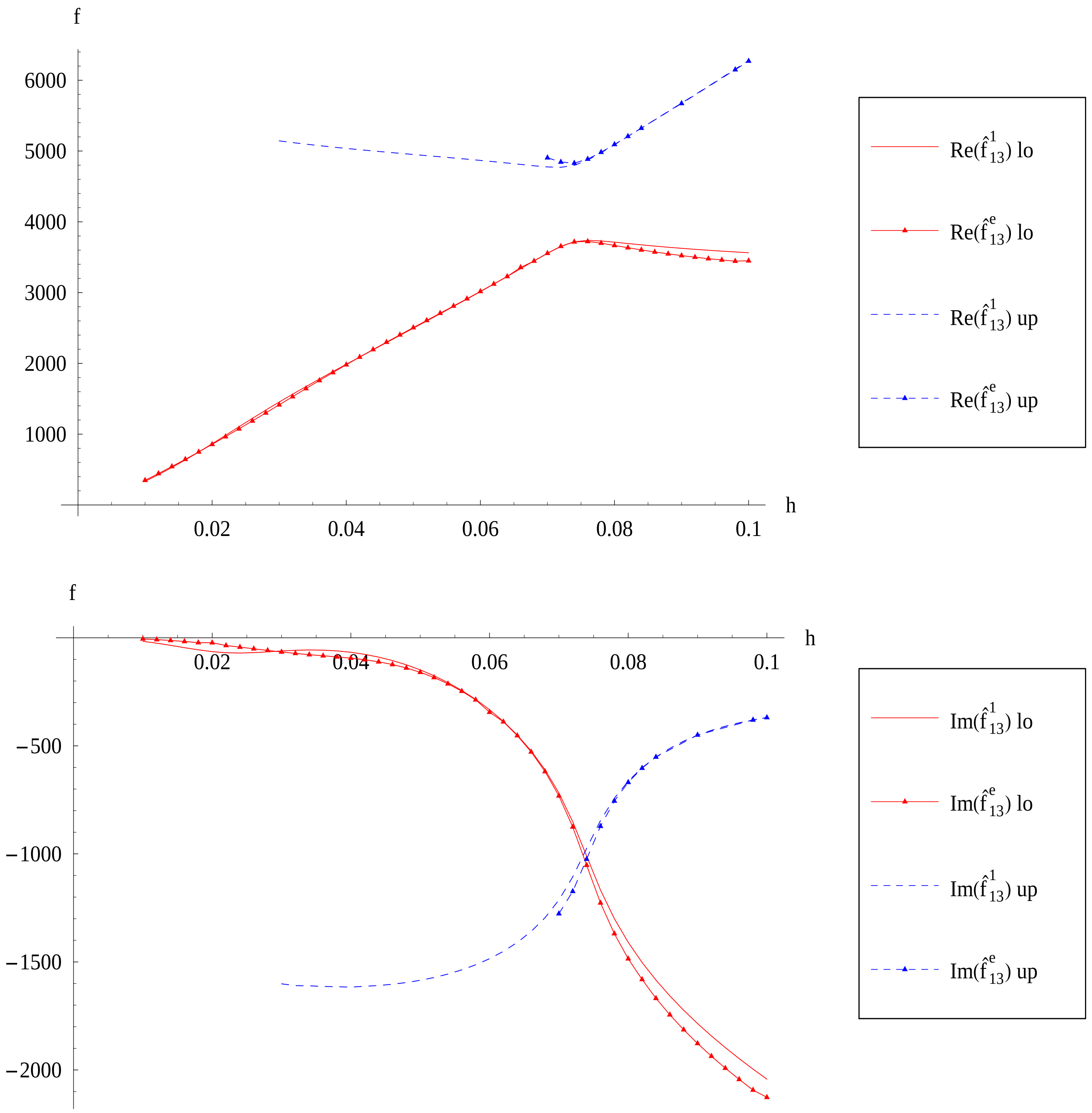

Figure 2: $1m \times 0.7m$ steel plate in water: evolution of the real and imaginary parts with the thickness of the plate of the $(m = 1, n = 3)$ resonance frequency. Comparison of the exact computations $(\hat{f}^e_{13})$ with the modified approximation $(\hat{f}^1_{13})$. Top: real part, bottom: imaginary part

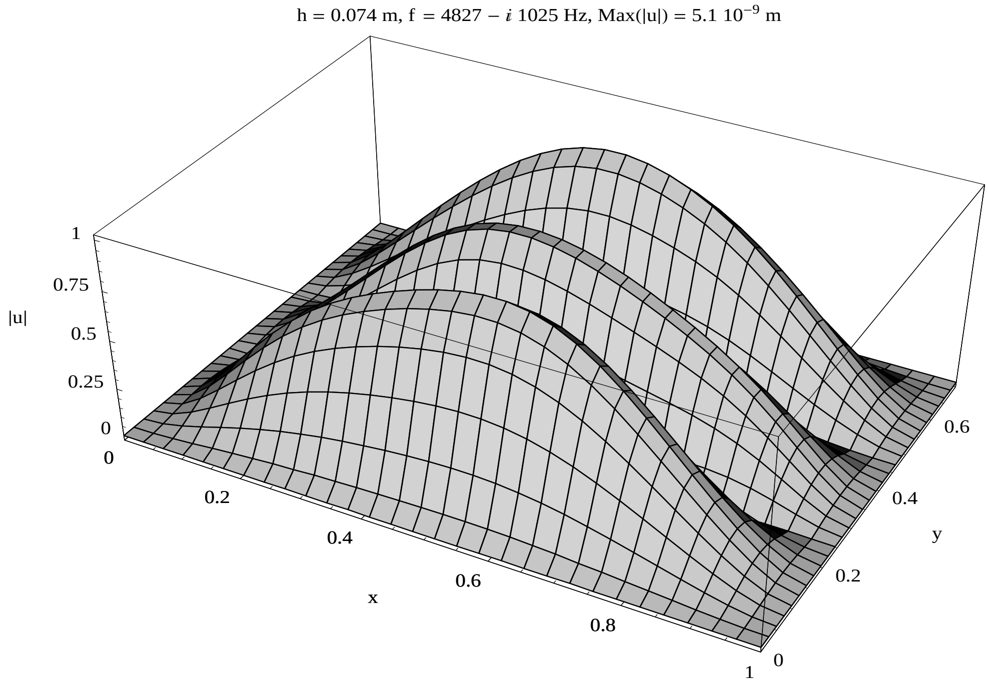

Figure 3: $1m \times 0.7m \times 0.074m$ steel plate in water: displacement of the plate (exact solution) close to the upper branch resonance frequency (excitation frequency $f = 4827 - \imath 1025\ Hz$)

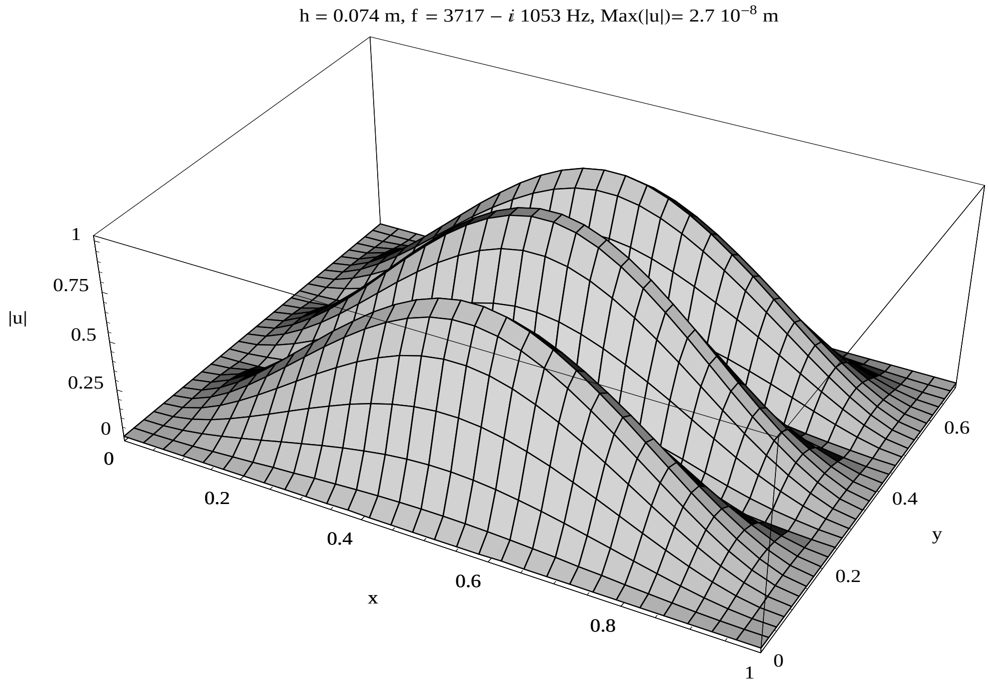

Figure 4: $1m \times 0.7m \times 0.074m$ steel plate in water: displacement of the plate (exact solution) close to the lower branch resonance frequency (excitation frequency $f = 3717 - \imath 1053\ Hz$)